

# On the $L_p$ -quantiles for the Student $t$ distribution

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May 26, 2016

## Abstract

$L_p$ -quantiles represent an important class of generalised quantiles and are defined as the minimisers of an expected asymmetric power function, see [Chen \(1996\)](#). For  $p = 1$  and  $p = 2$  they correspond respectively to the quantiles and the expectiles. In his paper [Koenker \(1993\)](#) showed that the  $\tau$  quantile and the  $\tau$  expectile coincide for every  $\tau \in (0, 1)$  for a class of rescaled Student  $t$  distributions with two degrees of freedom. Here, we extend this result proving that for the Student  $t$  distribution with  $p$  degrees of freedom, the  $\tau$  quantile and the  $\tau$   $L_p$ -quantile coincide for every  $\tau \in (0, 1)$  and the same holds for any affine transformation. Furthermore, we investigate the properties of  $L_p$ -quantiles and provide recursive equations for the truncated moments of the Student  $t$  distribution.

**keywords:** Expectiles; truncated moments; quantiles; risk measures; double factorial coefficient.

## 1 Introduction

During the last years the statistical, econometric and financial literature, has seen a growing interest in generalising the quantile tool to gather information about a random variable. Starting from the seminal paper by [Koenker and Bassett \(1978\)](#), quantile regression has emerged as a valid alternative to the mean regression in order to deeply investigate the relation between a response variable and some covariates, especially when the tails behaviour is of concern. Both the classical and Bayesian researches have developed new tools through the years, to solve the related inferential problems. We refer the interested reader to [Koenker \(2005\)](#) and the references therein for the classical point of view and to [Yu and Moyeed \(2001\)](#), [Taddy and Kottas \(2010\)](#), [Kottas and Krnjajić \(2009\)](#), [Bernardi et al. \(2015\)](#) and [Bernardi et al. \(2016\)](#) for the Bayesian one. In the same spirit, [Newey and Powell \(1987\)](#) and [Efron \(1991\)](#) proposed an alternative approach, namely the expectile regression, to summarise the conditional distribution of a dependent variable given the regressors. Expectile regression, based on the asymmetric least squares estimation, is a valid alternative to quantile regression and a natural extension of the mean regression. Although quantiles have an immediate interpretation, expectile regression methods have increased during the years (see for example [Schnabel and Eilers, 2009](#), [De Rossi and Harvey, 2009](#), [Sobotka and Kneib, 2012](#), [Guo et al., 2015](#) and [Sobotka et al., 2013](#)). One of the reasons resides in the generalisation of the mean regression and the easiness of calculation since these methods involve a minimisation problem which is continuously differentiable. Moreover, as shown in [Jones \(1994\)](#), expectiles are linked to quantiles, indeed they can be interpreted as quantiles of a related distribution.

More general tools, which include expectiles and quantiles, have been introduced by [Breckling and Chambers \(1988\)](#) and by [Chen \(1996\)](#) who proposed the use of  $M$ -quantiles and  $L_p$ -quantiles respectively. The  $M$ -quantiles extend the idea of quantiles by minimising a generic asymmetric loss function, while the  $L_p$ -quantiles minimise an asymmetric power function. The property of minimising a given loss function,

shared by all the generalised quantiles, finds its roots in decision theory (see for instance [Savage, 1971](#)) but was named elicibility by [Lambert et al. \(2008\)](#). Recently, elicitable statistical functionals have been investigated by [Gneiting \(2011\)](#) and [Heinrich \(2013\)](#). Generalised quantile regression models have also been considered in the literature, see, for example, [Chambers and Tzavidis \(2006\)](#) and [Pratesi et al. \(2009\)](#) for the  $M$ -quantile framework, and [Bernardi et al. \(2016\)](#) for a Bayesian approach to  $L_p$ -quantiles regression.

Quantiles and their generalisations play a central role also in the financial and actuarial science literature as key tools for computing capital requirements. The quantile of a loss distribution, named Value-at-Risk (VaR), was introduced by J.P. Morgan in 1994 to measure the riskiness of a financial position. Since then it has become the most widespread risk measure for regulatory purposes, see, for example, [Jorion \(2007\)](#). Despite its popularity the VaR, as a measure of risk, presents some important drawbacks. Firstly, it does not provide any information about losses exceeding the considered quantile; further it is not a sub-additive risk measure, meaning that it may not incentivise portfolio diversification. For these reasons several alternatives have been considered, one of which is the Expected Shortfall (ES) which is defined as the average of the quantiles exceeding the VaR. The debate on whether VaR or ES should be used for risk management purposes is still on going and was recently questioned in two consultative documents by the [Basel Committee on Banking Supervision \(2013\)](#) and the International Association of Insurance Supervisors [IAIS \(2014\)](#). Recently the debate on relevant properties that a risk measure should satisfy, like coherency and elicibility, (see [Artzner et al., 1999](#) and [Gneiting, 2011](#), respectively) brought the attention to other risk measures based on generalised quantiles. [Ziegel \(2014\)](#) and [Bellini and Bignozzi \(2015\)](#), for example, showed that the expectile is the unique risk measure that is both coherent and elicitable, while [Taylor \(2008\)](#) provided a methodology to estimate the ES and the VaR using expectiles. This explains the increased popularity of expectiles in the very recent quantitative risk management literature, see for instance [Bellini and Di Bernardino \(2015\)](#), [Jakobsons and Vanduffel \(2015\)](#) and the references therein. Since  $L_p$ -quantiles are natural extensions of expectiles and quantiles they received particular consideration in risk management as possible competitors of VaR; in particular, they are elicitable and have other interesting properties in terms of risk measures (see for instance [Bellini et al., 2014](#)).  $L_p$ -quantiles are also well known in actuarial science where they belong to the class of zero utility premiums (see for example [Deprez and Gerber, 1985](#)), while in financial mathematics they are considered as shortfall risk measures, see for instance [Föllmer and Schied \(2002\)](#). We refer the interested reader to [Embrechts et al. \(2014\)](#) and [Emmer et al. \(2015\)](#) for more detailed summaries on the recent developments on risk measurement.

In the present contribution we focus on the class of  $L_p$ -quantiles and express them in terms of moments and truncated moments. Thanks to this characterisation and to their definition as minimisers of a power loss function we are able to prove that for a Student  $t$  distribution with  $p$  degrees of freedom the  $\tau$  quantile and the  $\tau$   $L_p$ -quantile coincide for every  $\tau \in (0, 1)$  and the same holds for any affine transformation. We expect that this property of the Student  $t$  distribution may be useful for the explicit computation of its quantiles which are not available in closed form. Moreover, we investigate the properties of  $L_p$ -quantiles and provide recursive equations for the truncated moments of the Student  $t$  distribution.

Our results can be seen as an extension of the well known result proposed in [Koenker \(1992\)](#) and [Koenker \(1993\)](#) who showed that the  $\tau$  quantile and the  $\tau$  expectile coincide for every  $\tau \in (0, 1)$  for a particular class of distributions which corresponds to rescaled Student  $t$  distributions with two degrees of freedom. Some extensions of these results have been already considered recently by [Zou \(2014\)](#) who provided a class of distributions whose  $\omega(\tau)$  expectile coincide with the  $\tau$  quantile for any monotone function  $\omega(\cdot)$ .

The rest of the paper is organised as follows. In [Section 2](#) we discuss some properties of the  $L_p$ -quantiles and provide their representation in terms of moments and truncated moments. [Section 3](#) gathers general results on the Student  $t$  distribution and presents a recursive formula for the calculation of its truncated moments. Finally, [Section 4](#) contains the main results of the paper on the equality of quantiles and  $L_p$ -quantiles for a Student  $t$  distribution. The proof is rather long and it involves some ideas from combinatorial analysis therefore some technical details are postponed to the Appendix.

## 2 $L_p$ -quantiles

The  $L_p$ -quantile of a random variable  $Y$  at level  $\tau \in (0, 1)$ , denoted by  $\rho_{p,\tau}(Y)$ , is defined as:

$$\rho_{p,\tau}(Y) = \arg \min_{m \in \mathbb{R}} E[\tau((Y - m)_+)^p + (1 - \tau)((Y - m)_-)^p], \quad \text{for } p = 1, 2, \dots \quad (1)$$

where  $x_+ = \max(x, 0)$  and  $x_- = \max(-x, 0)$ , i.e., it is the minimiser of the asymmetric power function  $\tau((y - m)_+)^p + (1 - \tau)((y - m)_-)^p$ , provided that the expectation exists. It is easy to see that when  $p = 1$ ,  $\rho_{1,\tau}(Y)$  corresponds to the quantile  $q_\tau(Y)$  of  $Y$  and Equation (1) has a unique solution only if the distribution of  $Y$  is strictly increasing in a neighbourhood of  $\tau$ . When  $p = 2$ ,  $\rho_{2,\tau}(Y)$  corresponds to the  $\tau$  level expectile already introduced by Newey and Powell (1987). Moreover for  $p \geq 2$ , using the first order condition,  $L_p$ -quantiles can alternatively be written as the unique solution to the following equation

$$\tau E[(Y - m)_+^{p-1}] = (1 - \tau) E[(Y - m)_-^{p-1}], \quad \text{for all } \tau \in (0, 1). \quad (2)$$

From now on we only consider the case  $p \geq 2$  unless differently specified.

In the following proposition we summarise some properties of the  $L_p$ -quantiles that may be useful in the regression framework (see for instance Bernardi et al., 2016) or for risk measurement purposes (see for example Bellini et al., 2014).

**Proposition 1.** *Let  $X, Y$  be random variables with finite moment of order  $p-1$ . Then, for any  $\tau \in (0, 1)$  the following properties are satisfied:*

1. *Monotonicity:* If  $X \leq Y$  a.s., then  $\rho_{p,\tau}(X) \leq \rho_{p,\tau}(Y)$ ;
2. *Translation Invariance:* For any  $a \in \mathbb{R}$ ,  $\rho_{p,\tau}(X + a) = \rho_{p,\tau}(X) + a$ ;
3. *Positive homogeneity:* For any  $\lambda \geq 0$ ,  $\rho_{p,\tau}(\lambda X) = \lambda \rho_{p,\tau}(X)$ ;
4. *Symmetry:*  $\rho_{p,\tau}(X) = -\rho_{p,1-\tau}(-X)$ ;
5. *Law-invariance:* If  $X$  and  $Y$  have the same probability distribution then  $\rho_{p,\tau}(X) = \rho_{p,\tau}(Y)$ ;
6. *Concavity:* For any  $\beta \in [0, 1]$ ,  $\rho_{p,\tau}(\beta X + (1 - \beta)Y) \geq \beta \rho_{p,\tau}(X) + (1 - \beta) \rho_{p,\tau}(Y)$  if and only if  $p = 2$  and  $\tau \leq 1/2$ .

*Proof.* Properties 1-5 follow easily from the definition of  $L_p$ -quantiles, hence we only discuss the concavity one. From Equation (2),  $L_p$ -quantiles can be written as the unique solution to the equation  $E[\varphi(Y - m)] = 0$ , where the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$\varphi(t) = \tau(t_+)^{p-1} - (1 - \tau)(t_-)^{p-1}, \quad \text{for } \tau \in (0, 1).$$

From Proposition 6(b) in Bellini et al. (2014),  $L_p$ -quantiles are concave if and only if  $\varphi$  is concave. It is easy to see that expectiles, i.e. the  $L_p$ -quantiles when  $p = 2$ , are concave for  $\tau \leq 1/2$ . For  $p \geq 3$ , by computing the second derivative of  $\varphi$ , we note that  $\varphi$  is concave for  $t < 0$  and convex otherwise and never concave on the entire domain for any  $\tau$ .  $\square$

In the next proposition, we provide a further characterisation of the  $L_p$ -quantiles of a random variable  $Y$  with cumulative distribution function  $F_Y$  in terms of the raw moments  $E[Y^k]$  and the truncated  $k$ -th moments  $G_{k,Y}$

$$G_{k,Y}(x) = \int_{-\infty}^x y^k dF_Y(y), \quad x \in \mathbb{R},$$

for  $k = 0, \dots, p-1$ . This representation will be exploited throughout the paper to get the main results.

**Proposition 2.** *Let  $p$  be an odd number, then the  $L_p$ -quantile can be written as the unique solution of the following equation with respect to  $m$ :*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-m)^k (\tau E[Y^{p-1-k}] - G_{p-1-k,Y}(m)) = 0. \quad (3)$$

Similarly, for  $p$  even, the  $L_p$ -quantile can be written as:

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-m)^k (\tau E[Y^{p-1-k}] + (1-2\tau)G_{p-1-k,Y}(m)) = 0. \quad (4)$$

*Proof.* From Equation (2) and for any  $\tau \in (0, 1)$ :

$$\tau \int_m^\infty (y-m)^{p-1} dF_Y(y) = (1-\tau) \int_{-\infty}^m (-1)^{p-1} (y-m)^{p-1} dF_Y(y). \quad (5)$$

Let  $p$  be an odd number. Considering that

$$\tau \int_{-\infty}^\infty (y-m)^{p-1} dF_Y(y) = \int_{-\infty}^m (y-m)^{p-1} dF_Y(y) \quad (6)$$

and using the binomial expansion, we get

$$\tau \left( \int_{-\infty}^\infty \sum_{k=0}^{p-1} \binom{p-1}{k} y^{p-1-k} (-m)^k dF_Y(y) \right) = \left( \int_{-\infty}^m \sum_{k=0}^{p-1} \binom{p-1}{k} y^{p-1-k} (-m)^k dF_Y(y) \right).$$

The linearity of the integral gives

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-m)^k (\tau E[Y^{p-1-k}] - G_{p-1-k,Y}(m)) = 0. \quad (7)$$

Now let  $p$  be an even number; rearranging the terms in Equation (5) and adding and subtracting  $\int_{-\infty}^m (y-m)^{p-1} dF_Y(y)$ , leads to

$$\tau \left( \int_{-\infty}^\infty (y-m)^{p-1} dF_Y(y) - 2 \int_{-\infty}^m (y-m)^{p-1} dF_Y(y) \right) = \left( \int_{-\infty}^m -(y-m)^{p-1} dF_Y(y) \right).$$

Using again the binomial expansion, the equation becomes

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-m)^k (\tau E[Y^{p-1-k}] + (1-2\tau)G_{p-1-k,Y}(m)) = 0.$$

□

### 3 Some useful properties of the Student $t$ distribution

In this section we gather some results on the Student  $t$  distribution that will be used to prove the main theorems of the paper. Let  $Y$  be a Student  $t$  random variable with  $p$  degrees of freedom having density function

$$f_Y(y) = C_p \left( 1 + \frac{y^2}{p} \right)^{-\frac{p+1}{2}}, \quad \text{where } y \in \mathbb{R}, \quad C_p = \frac{\Gamma(\frac{p+1}{2})}{\sqrt{p\pi} \Gamma(\frac{p}{2})}.$$

When  $p$  is even  $C_p$  can be written as  $C_p = (p-1)!! p^{-1/2} / (2(p-2)!!)$  where the symbol  $!!$  denotes the double factorial defined as

$$x!! = \begin{cases} x \cdot (x-2) \cdot \dots \cdot 1 & \text{for } x \text{ positive odd integer,} \\ x \cdot (x-2) \cdot \dots \cdot 2 & \text{for } x \text{ positive even integer,} \end{cases}$$

with the convention that  $0!! = 0$ . The raw moments of order  $j$ ,  $1 \leq j \leq p-1$  are defined as

$$E[Y^j] = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \frac{\Gamma(\frac{j+1}{2}) \Gamma(\frac{p-j}{2})}{\sqrt{\pi} \Gamma(\frac{p}{2})} p^{\frac{j}{2}} = \prod_{k=1}^{j/2} \frac{2k-1}{p-2k} p & \text{if } j \text{ is even.} \end{cases}$$

In the following proposition we provide an easy way to compute the truncated moments of order  $j$ ,  $G_{j,Y}$ . We recall that the symbol  $\lfloor \cdot \rfloor$  denotes the lower integer part of a number.

**Proposition 3.** *Let  $Y$  be a Student  $t$  random variable with  $p$  degrees of freedom then the truncated moments of order  $j$ , for  $0 \leq j \leq p-1$ , are given by:*

$$G_{j,Y}(m) = -C_p \left(1 + \frac{m^2}{p}\right)^{\frac{1-p}{2}} \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} m^{j-1-2i} p^{i+1} \frac{(j-1)!!}{(j-1-2i)!!} \prod_{k=1}^{i+1} \frac{1}{p-j-2+2k} + F_Y(m) E[Y^j],$$

for  $0 < j \leq p-1$  and  $G_{0,Y}(m) = F_Y(m)$ .

*Proof.* It is immediate to see that

$$G_{0,Y}(m) = F_Y(m) \quad \text{and} \quad G_{1,Y}(m) = \frac{p}{1-p} C_p K_p,$$

where  $K_p = (1 + m^2/p)^{(1-p)/2}$ . For  $j \geq 2$ , formulas 2.147(2) and 2.263(1) in [Gradshteyn and Ryzhik \(2007\)](#), provide (for  $p$  odd and even respectively) a characterisation of the truncated moment of order  $j$  in terms of the truncated moment of order  $j-2$ :

$$G_{j,Y}(m) = -\frac{m^{j-1}}{p-j} p C_p K_p + \frac{p(j-1)}{p-j} G_{j-2,Y}(m).$$

By working backwards recursively, we obtain

$$G_{j,Y}(m) = -C_p \left(1 + \frac{m^2}{p}\right)^{\frac{1-p}{2}} \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} m^{j-1-2i} p^{i+1} \frac{(j-1)!!}{(j-1-2i)!!} \prod_{k=1}^{i+1} \frac{1}{p-j-2+2k} + F_Y(m) E[Y^j],$$

where we used that  $E[Y^j] = 0$  for  $j$  odd. □

From this expression it easy to see that the truncated moment of order  $p-1-k$ , evaluated at the  $\tau$ -quantile,  $\tau \in (0, 1)$ , that is in  $m = q_\tau(Y)$ , is:

$$G_{p-1-k,Y}(m) = -C_p K_p \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} m^{p-k-2-2i} p^{i+1} \frac{(p-k-2)!!}{(p-k-2-2i)!!} \prod_{j=1}^{i+1} \frac{1}{k-1+2j} + \tau E[Y^{p-1-k}] \quad (8)$$

for  $0 \leq k < p-1$ , and  $G_{0,Y}(m) = \tau$ .

Another interesting property of the Student  $t$  distribution, which will be extensively used to prove our results, is related to its cumulative distribution function. In what follows we report an interesting formula provided by [Shaw \(2005\)](#) which states that the cumulative distribution function  $F_Y$  for a Student  $t$  with even degrees of freedom  $p$  can be written as:

$$F_Y(m) = m K_p \left( \sum_{i=0}^{\frac{p}{2}-1} m^{2i} a_p(i) \right) + \frac{1}{2}, \quad (9)$$

where

$$a_p(i) = C_p \prod_{j=0}^i \frac{p-2j}{(2j+1)p}.$$

By denoting  $A_{p-1}$  the double factorial binomial coefficient

$$A_{p-1}(k) = \frac{(p-1)!!}{k!!(p-1-k)!!}$$

having the property  $A_{p-1}(k) = A_{p-1}(p-1-k)$ , the equality  $F_Y(m) = \tau$  can be written as

$$m K_p \left( \sum_{i=0}^{\frac{p}{2}-1} m^{2i} \frac{1}{2} A_{p-1}(2i+1) p^{-\frac{2i+1}{2}} \right) = \tau - \frac{1}{2}. \quad (10)$$

Equation (10) can be rearranged to:

$$\frac{2}{K_p} \left( \tau - \frac{1}{2} \right) = \sum_{i=0}^{\frac{p}{2}-1} m^{2i+1} p^{-\frac{2i+1}{2}} A_{p-1}(2i+1). \quad (11)$$

By squaring it and by changing the index of the summation with  $k = p/2 - 1 - i$  it can be rewritten as

$$\tau(1 - \tau) = \frac{1}{4} - \frac{K_p^2 p^{1-p}}{4} \left( \sum_{k=0}^{\frac{p}{2}-1} m^{p-1-2k} p^k A_{p-1}(2k) \right)^2. \quad (12)$$

## 4 $L_p$ -quantiles for the Student $t$ distribution

In his paper [Koenker \(1993\)](#) investigates the relation between quantiles and expectiles. In details he shows that for a particular class of distributions (that are affine transformations of a Student  $t$  random variable with two degrees of freedom) the  $\tau$  expectiles and the  $\tau$  quantiles coincide for every  $\tau$ . After that [Abdous and Rémillard \(1995\)](#) give sufficient conditions under which a quantile and an expectile coincide. Those results were then extended by [Zou \(2014\)](#) that analytically characterised distributions whose  $\omega(\beta)$  expectile and  $\beta$  quantile coincide, for any monotone function  $\omega(\cdot)$ . In what follows we extend all of those results and prove that the Student  $t$  distribution with  $p$  degrees of freedom has the peculiarity of having the  $L_p$ -quantile and the quantile coinciding for every  $\tau \in (0, 1)$ . This result is then extended to any affine transformation of the Student  $t$  random variable.

We first consider the case where  $p$  is an odd number.

**Theorem 4.1.** *Let  $p$  be an odd number and  $Y$  a random variable with Student  $t$  distribution with  $p$  degrees of freedom. Then the  $L_p$ -quantile  $\rho_{p,\tau}(Y)$  and the quantile  $q_\tau(Y)$  coincide for every  $\tau \in (0, 1)$ .*

*Proof.* As already said, when  $p = 1$  we have  $\rho_{1,\tau}(Y) = q_\tau(Y)$ , hence we consider the case  $p \geq 3$ . If the  $L_p$ -quantile and the quantile coincide then the equation

$$\sum_{k=0}^{p-1} B_{p-1}(k) (-m)^k (\tau E[Y^{p-1-k}] - G_{p-1-k,Y}(m)) = 0 \quad (13)$$

is satisfied for  $m = q_\tau(Y) = \rho_{p,\tau}(Y)$  where  $B_{p-1}(k) = \binom{p-1}{k} = \frac{(p-1)!}{k!(p-1-k)!}$ . Hence from now on we assume  $m = q_\tau(Y)$ . Using Equation (8) to write  $G_{p-1-k,Y}(m)$  in (13) we can cancel the term  $\tau E[Y^{p-1-k}]$  so that the equation to verify becomes

$$\sum_{k=0}^{p-2} B_{p-1}(k) (-m)^k \left( C_p K_p \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} m^{p-k-2-2i} p^{i+1} \frac{(p-k-2)!!}{(p-k-2-2i)!!} \prod_{j=1}^{i+1} \frac{1}{k-1+2j} \right) = 0, \quad (14)$$

which, using the properties of the binomial coefficients, can be rewritten as

$$K_p \frac{p^{\frac{1}{2}}}{\pi} \sum_{k=0}^{p-2} \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} (-1)^k m^{p-2-2i} p^i A_{p-1}(k) A_{p-1}(k+1+2i) = 0. \quad (15)$$

By dividing both sides for  $K_p p^{1/2}/\pi$  and inverting the order of the summations, we obtain

$$\sum_{i=0}^{\frac{p-3}{2}} m^{p-2-2i} p^i \sum_{k=0}^{p-2(i+1)} (-1)^k A_{p-1}(k) A_{p-1}(k+1+2i) = 0. \quad (16)$$

Equation (16) is a polynomial equation of order  $p-2$  containing only odd powers of  $m$ . The proof is concluded by showing that the coefficients of these powers are all null. Indeed the function

$$f_i(k) := (-1)^k A_{p-1}(k) A_{p-1}(k+1+2i), \quad \text{for } i = 0, \dots, \frac{p-3}{2}$$

satisfies  $f_i(k) = -f_i(p - 2(i + 1) - k)$  for  $k = 0, \dots, p - 2(i + 1)$ , hence all the terms in the second summation in Equation (16) cancel pairwise, the first with the last one, the second with the second to last and so on.  $\square$

The following theorem proves the analogous result for  $p$  even.

**Theorem 4.2.** *Let  $p$  be an even number and  $Y$  a random variable with Student  $t$  distribution with  $p$  degrees of freedom. Then the  $L_p$ -quantile  $\rho_{p,\tau}(Y)$  and the quantile  $q_\tau(Y)$  coincide for every  $\tau \in (0, 1)$ .*

*Proof.* If the  $L_p$ -quantile and the quantile coincide then the equation

$$\sum_{k=0}^{p-1} B_{p-1}(k)(-m)^k (\tau E[Y^{p-1-k}] + (1 - 2\tau)G_{p-1-k,Y}(m)) = 0, \quad (17)$$

is satisfied by  $m = q_\tau(Y) = \rho_{p,\tau}(Y)$ , for any  $\tau \in (0, 1)$ . For  $\tau = 1/2$  the result is immediate, hence we only consider  $\tau \in (0, 1) \setminus \{1/2\}$ .

Using Equation (8), the equation to solve becomes

$$\begin{aligned} (2\tau - 1) \sum_{k=0}^{p-2} B_{p-1}(k)(-m)^k \left( C_p K_p \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} m^{p-k-2-2i} p^{i+1} \frac{(p-k-2)!!}{(p-k-2-2i)!!} \prod_{j=1}^{i+1} \frac{1}{k-1+2j} \right) \\ + 2\tau(1 - \tau) \sum_{k=0}^{p-1} B_{p-1}(k)(-m)^k E[Y^{p-1-k}] = 0. \end{aligned} \quad (18)$$

For  $k = 0, \dots, p - 2$ ,

$$E[Y^{p-1-k}] = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \prod_{j=1}^{\frac{p-1-k}{2}} \frac{2j-1}{p-2j} p & \text{if } k \text{ is odd,} \end{cases}$$

thus using the properties of the factorial, Equation (18) can be rewritten as

$$\begin{aligned} \left( \tau - \frac{1}{2} \right) \frac{K_p}{2} \sum_{k=0}^{p-2} (-1)^k \left( \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} m^{p-2-2i} p^{i+\frac{1}{2}} A_{p-1}(k) A_{p-1}(k+1+2i) \right) \\ = \tau(1 - \tau) p^{\frac{p-1}{2}} \sum_{k=1, k \text{ odd}}^{p-1} m^k p^{-\frac{k}{2}} A_{p-1}(k). \end{aligned}$$

By applying the equalities (11) and (12) the equation to solve becomes

$$\begin{aligned} K_p^2 \sum_{k=0}^{p-2} (-1)^k \left( \sum_{i=0}^{\lfloor \frac{p-k-2}{2} \rfloor} m^{p-2-2i} p^{-\frac{p-2-2i}{2}} A_{p-1}(k) A_{p-1}(k+1+2i) \right) \\ = 1 - K_p^2 p^{1-p} \left( \sum_{k=0}^{\frac{p}{2}-1} m^{p-1-2k} p^k A_{p-1}(2k) \right)^2. \end{aligned}$$

Inverting the order of the two summations in the left hand side of the above equation, and rearranging the terms, we obtain

$$\begin{aligned} \sum_{i=0}^{\frac{p}{2}-1} m^{-2i} p^i \sum_{k=0}^{p-2(i+1)} (-1)^k A_{p-1}(k) A_{p-1}(k+1+2i) + \left( \sum_{k=0}^{\frac{p}{2}-1} m^{\frac{p}{2}-2k} p^{-\frac{p}{4}+k} A_{p-1}(2k) \right)^2 \\ = K_p^{-2} m^{2-p} p^{\frac{p}{2}-1}. \end{aligned} \quad (19)$$

For  $p = 2$ , the equation in (19) is easily verified, hence we only consider the case  $p \geq 4$ . Using the fact that

$$\left( \sum_{k=0}^{\frac{p}{2}-1} m^{\frac{p}{2}-2k} p^{-\frac{p}{4}+k} A_{p-1}(2k) \right)^2 = \quad (20)$$

$$\sum_{j=0}^{\frac{p}{2}-2} m^{-2j} p^j \sum_{\substack{h=1, \\ h \text{ odd}}}^{p-2j-3} A(h)A(h+2j+1) + \sum_{j=1}^{\frac{p}{2}} m^{2j} p^{-j} \sum_{\substack{h=0, \\ h \text{ even}}}^{p-2j} A(h)A(h+2j-1) \quad (21)$$

(for the proof of this result see the Appendix), it follows that Equation (19) becomes

$$\begin{aligned} & \sum_{i=0}^{\frac{p}{2}-1} m^{-2i} p^i \sum_{\substack{k=0, \\ k \text{ even}}}^{p-2(i+1)} A_{p-1}(k)A_{p-1}(k+1+2i) + \sum_{j=1}^{\frac{p}{2}} m^{2j} p^{-j} \sum_{\substack{h=0, \\ h \text{ even}}}^{p-2j} A(h)A(h+2j-1) \\ &= K_p^{-2} m^{2-p} p^{\frac{p}{2}-1}. \end{aligned} \quad (22)$$

Using formula (27) in Gould and Quaintance (2012), for which given an even number  $l$  we have  $A_{p-1}(l) = B_{\frac{p-1}{2}}(l/2)$  the summations for  $k$  and  $h$  even can be rewritten respectively as

$$\sum_{\substack{k=0, \\ k \text{ even}}}^{p-2(i+1)} A_{p-1}(k)A_{p-1}(k+1+2i) = \sum_{j=0}^{\frac{p}{2}-(i+1)} B_{\frac{p-1}{2}}(j)B_{\frac{p-1}{2}}\left(\frac{p}{2}-(i+1)-j\right) = B_{p-1}\left(\frac{p}{2}-1-i\right),$$

and

$$\sum_{\substack{h=0, \\ h \text{ even}}}^{p-2j} A_{p-1}(h)A_{p-1}(h+2j-1) = \sum_{h=0}^{\frac{p}{2}-j} B_{\frac{p-1}{2}}(h)B_{\frac{p-1}{2}}\left(\frac{p}{2}-j-h\right) = B_{p-1}\left(\frac{p}{2}-j\right),$$

where the last equality is obtained using the Chu-Vandermonde binomial identity, see for instance Askey (1975), pages 59-60. It follows that Equation (19) becomes

$$\begin{aligned} & \sum_{i=0}^{\frac{p}{2}-1} m^{-2i} p^i B_{p-1}\left(\frac{p}{2}-1-i\right) + \sum_{j=1}^{\frac{p}{2}} m^{2j} p^{-j} B_{p-1}\left(\frac{p}{2}-1+j\right) \\ &= K_p^{-2} m^{2-p} p^{\frac{p}{2}-1}. \end{aligned} \quad (23)$$

The proof is concluded by decomposing the term  $K_p^{-2} = (1 + m^2/p)^{p-1}$  using the binomial expansion

$$K_p^{-2} p^{\frac{p}{2}-1} m^{2-p} = \sum_{i=0}^{p-1} B_{p-1}(i) m^{-p+2+2i} p^{\frac{p}{2}-1-i} = \sum_{i=-\frac{p}{2}}^{\frac{p}{2}-1} m^{-2i} p^i B_{p-1}\left(\frac{p}{2}-1-i\right).$$

□

**Corollary 1.** *Let  $X$  be any affine transformation of a random variable with Student  $t$  distribution with  $p$  degrees of freedom, that is  $X \stackrel{d}{=} a + bY$ ,  $a \in \mathbb{R}$ ,  $b > 0$ , then  $q_\tau(X) = \rho_{p,\tau}(X)$ .*

*Proof.* The proof follows immediately from the translation invariance and positive homogeneity of the  $L_p$ -quantiles. □



## Appendix

Let us take a closer look at the squared term in Equation (19)

$$\left( \sum_{k=0}^{\frac{p}{2}-1} m^{\frac{p}{2}-2k} p^{-\frac{p}{4}+k} A_{p-1}(2k) \right)^2 = \left( \sum_{k=0}^{\frac{p}{2}-1} m^{p-4k} p^{-\frac{p}{2}+2k} A_{p-1}(2k)^2 \right) + 2 \sum_{k=0}^{\frac{p}{2}-2} \sum_{j=k+1}^{\frac{p}{2}-1} m^{\frac{p}{2}-2(k+j)} p^{-\frac{p}{2}+k+j} A_{p-1}(2k) A_{p-1}(2j) = \quad (24)$$

$$m^p p^{-\frac{p}{2}} + m^{-p+4} p^{\frac{p}{2}-2} A(1)^2 + \sum_{i=1}^{p-3} m^{p-2i} p^{-\frac{p}{2}+i} \left( A(i)^2 \mathbb{I}_{\{i \text{ even}\}} + \sum_{\substack{h=\max\{0, 2i+2-p\}, \\ h \text{ even}}}^{i-1} 2A(h)A(2i-h) \right) =$$

$$m^p p^{-\frac{p}{2}} + m^{-p+4} p^{\frac{p}{2}-2} A(1)^2 + \sum_{j=1-\frac{p}{2}}^{\frac{p}{2}-3} m^{-2j} p^j \left( A(j + \frac{p}{2})^2 \mathbb{I}_{\{j+\frac{p}{2} \text{ even}\}} + \sum_{\substack{h=\max\{0, 2j+2\}, \\ h \text{ even}}}^{\frac{p}{2}+j-1} 2A(h)A(2j+p-h) \right) =$$

$$m^{-p+4} p^{\frac{p}{2}-2} A(1)^2 + \sum_{j=0}^{\frac{p}{2}-3} m^{-2j} p^j \left( A(j + \frac{p}{2})^2 \mathbb{I}_{\{j+\frac{p}{2} \text{ even}\}} + \sum_{\substack{k=0, \\ k \text{ even}}}^{\frac{p}{2}-j-3} 2A(k+2j+2)A(p-k-2) \right) \quad (25)$$

$$+ m^p p^{-\frac{p}{2}} + \sum_{j=1}^{\frac{p}{2}-1} m^{2j} p^{-j} \left( A(\frac{p}{2} - j)^2 \mathbb{I}_{\{\frac{p}{2}-j \text{ even}\}} + \sum_{\substack{h=0, \\ h \text{ even}}}^{\frac{p}{2}-j-1} 2A(h)A(-2j+p-h) \right) =$$

$$m^{-p+4} p^{\frac{p}{2}-2} A(1)^2 + \sum_{j=0}^{\frac{p}{2}-3} m^{-2j} p^j \left( A(j + \frac{p}{2})^2 \mathbb{I}_{\{j+\frac{p}{2} \text{ even}\}} + \sum_{\substack{h=1, \\ h \text{ odd}}}^{\frac{p}{2}-j-2} 2A(h)A(h+2j+1) \right) \quad (26)$$

$$+ m^p p^{-\frac{p}{2}} + \sum_{j=1}^{\frac{p}{2}-1} m^{2j} p^{-j} \left( A(\frac{p}{2} - j)^2 \mathbb{I}_{\{\frac{p}{2}-j \text{ even}\}} + \sum_{\substack{h=0, \\ h \text{ even}}}^{\frac{p}{2}-j-1} 2A(h)A(h+2j-1) \right) =$$

$$\sum_{j=0}^{\frac{p}{2}-2} m^{-2j} p^j \sum_{\substack{h=1, \\ h \text{ odd}}}^{p-2j-3} A(h)A(h+2j+1) + \sum_{j=1}^{\frac{p}{2}} m^{2j} p^{-j} \sum_{\substack{h=0, \\ h \text{ even}}}^{p-2j} A(h)A(h+2j-1).$$

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